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THE TWO DIMENSIONAL CONTACT PROBLEM OF A ROUGH STAMP SLIDING SLOWLY ON AN ELASTIC LAYER—I. GENERAL CONSIDERATIONS AND THICK LAYER ASYMPTOTICS

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Abstract—An approximate solution is obtained for the contact problem of a layer of finite thickness loaded by a rough cylindrical stamp which moves along the boundary. The coefficient of friction is assumed to be constant. The lower side of the layer is attached to a rigid base. In the problem inertial forces are neglected and the solution is approximated by a plane strain solution. This solution is presented in the form of a (convergent) series expansion in powers of the reciprocal thickness parameter, i.e. the ratio of the values of the thickness and the length of the contact region. The coefficients in this expansion satisfy singular integral equations. Numerical results are obtained for large values of the thickness parameter. In part II of this investigation the thin layer asymptotics will be given.

1. INTRODUCTION

IN SOME recent papers [1, 2] we considered an elastic layer loaded in plane strain by rigid smooth stamps. The lower side of the layer was supposed to be attached to a rigid base or to slide without friction along the base, e.g. on a fluid film. In [1] we constructed an approximate solution for the problem of a cylindrical stamp on a layer with a very small thickness parameter, i.e. the ratio of the thickness of the layer and the width (or half width) of the contact region. Dealing with a rectangular block loaded by a force and a moment [2] we obtained approximate solutions both for the thick and for the thin layer. In the papers [1] and [2] we treated compressible and incompressible layers separately. We did not consider the thick layer asymptotics for the cylindrical stamp in [1], as this problem has been treated adequately in the literature [3, 4].

In this paper, consisting of two parts, we deal with a class of similar problems which likewise may be discussed within the framework of linear elasticity. We now assume that the stamp is rigid and rough, and slides slowly along the boundary of the layer. Obviously we only have to consider a layer attached to a fixed base. When dealing with thin layers we also limit our considerations to compressible material. The extension of the problem under discussion to some of the other boundary value problems that have been treated in [1] and [2] is obvious.

We shall assume that the friction coefficient is constant and that the motion is slow to justify the disregard of inertial forces.

In the first part of the paper we derive an integral equation for the pressure under the stamp. A rigorous analytical solution for this equation may be obtained in the form of a series expansion in powers of the reciprocal thickness parameter. It may be expected that this formal solution converges for all values of the thickness parameter larger than two.

However, for practical computations, only values larger than four can be taken into consideration. With the methods of part II very small values of the thickness parameter are studied. The approximate solution for intermediate values of the thickness parameter may be obtained by interpolation of the results of the two parts.

The expansion as a series of powers of the reciprocal thickness parameter involves coefficients which satisfy singular integral equations. These may be solved with the aid of the theory of functions of a complex variable [5]. The zero'th order term is the known half-plane solution [6] and the other terms represent the perturbations originating from the presence of the lower boundary.

An interesting feature of the solution is that all the displacements are bounded whereas the half-plane solution shows a normal displacement at the upper bounding line which becomes logarithmically infinite.

2. STATEMENT OF THE PROBLEM

We consider an isotropic homogeneous elastic layer, occupying the region of space $-\infty < \bar{x} < \infty$, $-b < \bar{y} < 0$, $-\infty < \bar{z} < \infty$, where $(\bar{x}, \bar{y}, \bar{z})$ is a right-handed cartesian coordinate system (cf. Fig. 1). The layer is loaded by a rough rigid cylinder of infinite extension in the $\bar{z}$-direction, with radius R. The cylinder slides along the boundary in the positive $\bar{x}$-direction with the velocity V. The (constant) coefficient of friction is f. The cylinder is pressed into the layer by a force P, measured per unit of length in the $\bar{z}$-direction and to maintain the uniform motion a horizontal force fP is applied in the positive $\bar{x}$-direction. At $\bar{y} = -b$, the layer is attached to a rigid base. Within linear elastodynamics the problem to be solved is formulated by the following system of equations

$$
\begin{aligned}
\mu\nabla^2 u + (\lambda+\mu)(u_{,\bar{x}} + v_{,\bar{y}})_{,\bar{x}} &= \rho\frac{\partial^2 u}{\partial t^2}, \\
\mu\nabla^2 v + (\lambda+\mu)(u_{,\bar{x}} + v_{,\bar{y}})_{,\bar{y}} &= \rho\frac{\partial^2 v}{\partial t^2},
\end{aligned}
\tag{2.1}
$$

where u and v are the displacements in the $\bar{x}$- and $\bar{y}$-directions, respectively, ∇^2 is the plane Laplacian operator, λ and μ are the Lamé constants, ρ is the density of the material of the

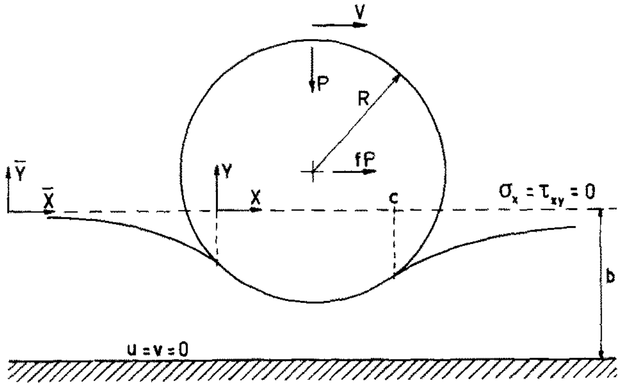

FIG. 1. Geometry of the problem.

layer, t is the time and $u,_{\bar{x}}$ denotes $\partial u/\partial \bar{x}$, etc. The solution of (2.1) has to satisfy some boundary conditions. Before formulating these we introduce a system of moving coordinates (x, y, z), fixed in the rigid body and defined by

$$x = \bar{x} - Vt, \qquad y = \bar{y}, \qquad z = \bar{z}. \tag{2.2}$$

In these coordinates (2.1) becomes if we confine ourselves to the quasistatic case ($\partial/\partial t = 0$)

$$\begin{aligned} \left(1-\frac{V^2}{c_1^2}\right)u,_{xx}+\left(\frac{c_2}{c_1}\right)^2 u,_{yy}+\left(1-\frac{c_2^2}{c_1^2}\right)v,_{xy} &= 0, \\ \left(1-\frac{V^2}{c_2^2}\right)v,_{xx}+\left(\frac{c_1}{c_2}\right)^2 v,_{yy}+\left(\frac{c_1^2}{c_2^2}-1\right)u,_{xy} &= 0, \end{aligned} \tag{2.3}$$

with

$$c_1^2 = \frac{\lambda+2\mu}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho}, \qquad (c_2 < c_1). \tag{2.4}$$

The following boundary conditions have to be satisfied at $y = 0$

$$v = v_0 + \frac{(x-d)^2}{2R}, \qquad (0 \le x \le c), \tag{2.5}$$

$$t_{xy} = -f\sigma_y, \qquad (0 \le x \le c), \tag{2.6}$$

$$\sigma_y = t_{xy} = 0, \qquad (x < 0; x > c), \tag{2.7}$$

where c is the length of the interval of contact, v_0 is the displacement at $x = 0$ and d is the x-coordinate of the point where the displacement has a horizontal tangent. The stresses are denoted by σ_x, σ_y and t_{xy}. We note that, assuming c to be prescribed, P, v_0 and d are unknown constants which are determined in the theory. We introduce the pressure p at $y = 0$ by

$$\sigma_y = -p(x), \tag{2.8}$$

which has to meet the inequality

$$p(x) \ge 0, \qquad (0 \le x \le c). \tag{2.9}$$

At $y = -b$ the boundary conditions are

$$u = v = 0, \qquad (-\infty < x < \infty). \tag{2.10}$$

The solution of the problem (2.3) with the boundary conditions (2.5)–(2.7) and (2.10) is strongly dependent on the parameters V/c_1, V/c_2 and we have to distinguish three different cases: $V < c_2$, $c_2 < V < c_1$ and $V > c_1$. If $V < c_2$, (2.3) is an elliptic system, while it is of the hyperbolic type if $V > c_2$. In this investigation we limit ourselves to the elliptic system with

$$V/c_2 \ll 1, \tag{2.11}$$

so that we may neglect the terms with $(V/c_1)^2$ and $(V/c_2)^2$ and we may replace (2.3) by the equations of elastostatics

$$\left.\begin{aligned} \nabla^2 u + \frac{1}{1-2\nu}(u,_x + v,_y),_x &= 0, \\ \nabla^2 v + \frac{1}{1-2\nu}(u,_x + v,_y),_y &= 0, \end{aligned}\right\} (0 < \nu < \tfrac{1}{2}), \tag{2.12}$$

where $\nu = \lambda/[2(\lambda+\mu)]$ is Poisson's ratio. We note that the method we use for the solution of (2.12) may equally be employed for the more general system of equations (2.3) with $V/c_2 < 1$, the only difference being the occurrence of an apparent anisotropy.

For the elliptic case the regularity conditions at infinity can be formulated as follows: u and v and their derivatives tend to zero as $|x| \to \infty$, for all values of y in the strip $-b \leq y \leq 0$.

The solution of the boundary problem (2.12), (2.5)–(2.7) and (2.10) may be obtained by transforming it into an integral equation with the aid of Fourier's integral formula. We have the relations

$$\bar{f}(\xi) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx, \qquad f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \bar{f}(\xi) e^{-i\xi x} \, d\xi, \tag{2.13}$$

which hold for suitable regular functions $f(x)$. Applying (2.13) to (2.12) and using the boundary conditions (2.5)–(2.7) and (2.10) we find

$$v(x, 0) = \frac{b}{\sqrt{(2\pi)}} \frac{1-\nu}{\mu} \int_{-\infty}^{\infty} \bar{p}(\xi) \left\{ \frac{K_1(\xi b)}{\xi b} + \frac{if}{2(1-\nu)} \frac{K_2(\xi b)}{\xi b} \right\} e^{-i\xi x} \, d\xi, \tag{2.14}$$

where $v(x, 0)$ is the normal displacement v at $y = 0$, and the functions $K_1(\xi b)$ and $K_2(\xi b)$ are given by

$$K_1(\xi b) = \frac{2\xi b - (3-4\nu) \sinh 2\xi b}{2\xi^2 b^2 + (3-4\nu) \cosh 2\xi b + (5 - 12\nu + 8\nu^2)}, \tag{2.15}$$

$$K_2(\xi b) = \frac{2\xi^2 b^2 - (3-4\nu)(1-2\nu)(\cosh 2\xi b - 1)}{2\xi^2 b^2 + (3-4\nu) \cosh 2\xi b + (5 - 12\nu + 8\nu^2)}, \tag{2.16}$$

respectively.

Application of the convolution theorem to (2.14) yields

$$v(x, 0) = \frac{1}{2\pi} \frac{1-\nu}{\mu} \int_{-\infty}^{\infty} p(y) \left\{ S_1(x-y) + \frac{f}{2(1-\nu)} S_2(x-y) \right\} dy, \tag{2.17}$$

with

$$S_1(t) = \int_{-\infty}^{\infty} \frac{\cos \xi t}{\xi} K_1(\xi b) \, d\xi, \tag{2.18}$$

$$S_2(t) = \int_{-\infty}^{\infty} \frac{\sin \xi t}{\xi} K_2(\xi b) \, d\xi. \tag{2.19}$$

We shall write the integral equation (2.17) in non-dimensional form. We introduce

$$x = x'c, \quad y = y'c, \quad b = qc, \quad v_0 = v_0'c, \quad R = R'c, \quad \xi = \xi'/c, \quad d = d'c, \quad p' = \frac{p(1-\nu)}{2\pi\mu}, \tag{2.20}$$

and then the equation (2.17) may be written as follows

$$\int_0^1 p(y)\{S_1(x-y)+\frac{f}{2(1-\nu)}S_2(x-y)\}\,\mathrm{d}y = \frac{(x-d)^2}{2R}+v_0, \qquad (0 \le x \le 1). \qquad (2.21)$$

In (2.21) we have omitted the primes and we have used the boundary conditions (2.5) and (2.7).

3. THE THICK PLATE

We consider the case

$$q \gg 1. \qquad (3.1)$$

The functions $S_1(t)$ and $S_2(t)$ can be expanded as uniformly convergent series

$$S_1(t) = 2\log\frac{|t|}{2q}+\sum_{k=0}^{\infty}\alpha_k\left(\frac{t}{2q}\right)^{2k}, \qquad (3.2)$$

$$S_2(t) = -(1-2\nu)\pi\,\mathrm{sign}\,t+\sum_{k=1}^{\infty}\beta_k\left(\frac{t}{2q}\right)^{2k-1}. \qquad (3.3)$$

We have evaluated the first four, respective three coefficients in the expansions (3.2) and (3.3) by numerical integration for several values of Poisson's ratio. The results are presented in Table 1.

TABLE 1

	α_0	α_1	α_2	α_3	β_1	β_2	β_3
$\nu = 0$	+2·131	−4·593	+5·577	−6·803	+5·993	−7·109	+9·43
0·2	+2·268	−5·176	+6·781	−8·646	+4·353	−6·308	+9·17
0·3	+2·440	−5·728	+7·846	−10·238	+3·640	−6·095	+9·31
0·4	+2·752	−6·623	+9·540	−12·767	+3·025	−6·076	+9·77
0·5	+3·339	−8·189	+12·531	−17·293	+2·551	−6·386	+10·84

After introducing (3.2) and (3.3) into (2.21) the integral equation takes the form

$$\begin{aligned} 2\int_0^1 p(y)\log|x-y|\,\mathrm{d}y &+ \sum_{k=1}^{\infty}\frac{\alpha_k}{(2q)^{2k}}\int_0^1 p(y)(x-y)^{2k}\,\mathrm{d}y \\ &+\frac{f}{2(1-\nu)}\sum_{k=1}^{\infty}\frac{\beta_k}{(2q)^{2k-1}}\int_0^1 p(y)(x-y)^{2k-1}\,\mathrm{d}y - g\int_0^x p(y)\,\mathrm{d}y \\ &+ g\int_x^1 p(y)\,\mathrm{d}y = \frac{(x-d)^2}{2R}+v_0-(\alpha_0-2\log 2q)P, \end{aligned} \qquad (3.4)$$

with

$$g = \frac{1-2\nu}{2(1-\nu)}f\pi, \qquad (3.5)$$

and

$$P = \int_0^1 p(y)\,\mathrm{d}y, \tag{3.6}$$

where P is the total force per unit length, measured in the unit $2\pi\mu c/(1-\nu)$. We differentiate (3.4) with respect to x and obtain the singular integral equation

$$\int_0^1 \frac{p(y)}{x-y}\,\mathrm{d}y - gp(x) = \frac{x-d}{2R} - \sum_{k=1}^{\infty} \frac{\alpha_k \,.\, k}{(2q)^{2k}} \int_0^1 p(y)(x-y)^{2k-1}\,\mathrm{d}y$$

$$-\frac{f}{2(1-\nu)} \sum_{k=1}^{\infty} \frac{\beta_k}{(2q)^{2k-1}} \frac{2k-1}{2} \int_0^1 p(y)(x-y)^{2k-2}\,\mathrm{d}y. \tag{3.7}$$

A direct way for solving (3.7) is to expand $p(y)$ and d in the following series

$$p(y) = \sum_{l=0}^{\infty} \frac{p_l(y)}{(2q)^l}, \tag{3.8}$$

$$d = \sum_{l=0}^{\infty} \frac{d_l}{(2q)^l}, \tag{3.9}$$

By equating equal powers of $(2d)^{-1}$ we obtain the system of integral equations

$$\int_0^1 \frac{p_l(y)}{y-x}\,\mathrm{d}y + gp(x) = -\frac{x}{2R}\delta_{0l} + \frac{d_l}{2R} + \sum_{k=1}^{\infty} \alpha_k k \int_0^1 p_{l-2k}(y)(x-y)^{2k-1}\,\mathrm{d}y$$

$$+\frac{f}{2(1-\nu)} \sum_{k=1}^{\infty} \beta_k \frac{2k-1}{2} \int_0^1 p_{l-2k}(y)(x-y)^{2k-2}\,\mathrm{d}y, \qquad (l = 0, 1, 2 \ldots), \tag{3.10}$$

where

$$p_s = 0, \quad \text{for } s < 0 \tag{3.11}$$

and δ_{0l} is the Kronecker delta.

The system (3.10) has to be solved step by step. Each of the equations is of the general form

$$\int_0^1 \frac{p_l(y)}{y-x}\,\mathrm{d}y + gp(x) = Q(x), \tag{3.12}$$

where $Q(x)$ is a polynomial.

4. DISCUSSION OF THE FUNDAMENTAL EQUATION (3.12)

The theory of the solution of (3.12) is well-known [5] and we shall not enter into the details of it. It appears that (3.12) has a solution, bounded both at $x = 0$ and at $x = 1$, only if the function $Q(x)$ satisfies the condition

$$\int_0^1 \frac{Q(t)}{t^{\theta}(1-t)^{1-\theta}}\,\mathrm{d}t = 0, \tag{4.1}$$

where θ is defined by

$$\theta = \frac{1}{\pi}\arctan\frac{2(1-\nu)}{f(1-2\nu)}, \qquad (0 < \theta < \tfrac{1}{2}). \tag{4.2}$$

In that case the bounded solution appears to be

$$p_l(x) = -\frac{\sin^2\pi\theta}{\pi^2}x^\theta(1-x)^{1-\theta}\int_0^1 \frac{Q(t)}{t^\theta(1-t)^{1-\theta}}\frac{dt}{t-x}+\frac{\sin 2\pi\theta}{2\pi}Q(x). \tag{4.3}$$

The integral in (4.3) can be evaluated by using a method given in [6]. We have calculated

$$\int_0^1 \frac{Q(t)}{t^\theta(1-t)^{1-\theta}}\frac{dt}{t-z} = \frac{\pi e^{\pi i\theta}}{\sin\pi\theta}\left[\frac{Q(z)}{z^\theta(1-z)^{1-\theta}} - \sum_{k=1}^{n} s_k z^k\right], \tag{4.4}$$

where the coefficients s_k are determined from the expansion

$$\frac{Q(t)}{t^\theta(1-t)^{1-\theta}} = s_n t^n + s_{n-1}t^{n-1} + \ldots + s_0 + \frac{s_{-1}}{t} + \ldots, \tag{4.5}$$

which holds for $|t| \to \infty$.

We note that we can always satisfy the equation (4.1) by adjusting the values of d_l.

5. APPLICATION

We have evaluated the first four functions $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_3(x)$ and the corresponding parameters d_0, d_1, d_2 and d_3. We found

$$p_0(x) = \frac{\sin\pi\theta}{\pi}\frac{1}{2R}x^\theta(1-x)^{1-\theta}, \qquad (>0), \tag{5.1}$$

$$p_1(x) = 0, \tag{5.2}$$

$$p_2(x) = -\frac{\alpha_1\theta(1-\theta)}{4R}\frac{\sin\pi\theta}{\pi}x^\theta(1-x)^{1-\theta}, \tag{5.3}$$

$$p_3(x) = -\frac{\beta_2\theta(1-\theta)f}{16R(1-\nu)}\frac{\sin\pi\theta}{\pi}x^\theta(1-x)^{1-\theta}[3x+(1-5\theta)], \tag{5.4}$$

together with

$$d_0 = 1-\theta \tag{5.5}$$

$$d_1 = -\frac{\beta_1\theta(1-\theta)f}{8(1-\nu)}, \tag{5.6}$$

$$d_2 = -\frac{\alpha_1}{3}\theta(1-\theta)(1-2\theta), \tag{5.7}$$

$$d_3 = \frac{f}{2(1-\nu)}\left[\frac{\alpha_1\beta_1}{8}\theta^2(1-\theta)^2 - \frac{3\beta_2}{16}\theta(1-\theta)(2-5\theta+5\theta^2)\right]. \tag{5.8}$$

Similar to (3.8) and (3.9) we introduce the following expansion for the total load

$$P = \sum_{l=0}^{\infty}\frac{P_l}{(2q)^l}, \tag{5.9}$$

with

$$P_l = \int_0^1 p_l(y)\,\mathrm{d}y. \tag{5.10}$$

Using (5.1)–(5.4) we find for the first four P_l's

$$P_0 = \frac{\theta(1-\theta)}{4R}, \tag{5.11}$$

$$P_1 = 0, \tag{5.12}$$

$$P_2 = -\frac{\alpha_1}{8R}\theta^2(1-\theta)^2, \tag{5.13}$$

$$P_3 = -\frac{\beta_2 f}{16R(1-\nu)}\theta^2(1-\theta)^2(1-2\theta). \tag{5.14}$$

We also evaluated the first approximations for the displacement v_0. From (3.4) we find

$$v_0 = (\alpha_0 - 2\log 2q)P - \frac{d^2}{2R} + 2\int_0^1 p(y)\log y\,\mathrm{d}y$$
$$+\sum_{k=1}^{\infty}\frac{\alpha_k}{(2q)^{2k}}\int_0^1 p(y)y^{2k}\,\mathrm{d}y - \frac{f}{2(1-\nu)}\sum_{k=1}^{\infty}\frac{\beta_k}{(2q)^{2k-1}}\int_0^1 p(y)y^{2k-1}\,\mathrm{d}y$$
$$+g\int_0^1 p(y)\,\mathrm{d}y. \tag{5.15}$$

Analogous to (5.9) we also expand v_0 in the series

$$v_0 = \sum_{l=0}^{\infty}\frac{v_{0l}}{(2q)^l} \tag{5.16}$$

and we find for the subsequent coefficients v_{0l}

$$v_{00} = (\alpha_0 - 2\log 2q + \pi\cot\pi\theta)\frac{\theta(1-\theta)}{4R} - \frac{(1-\theta)^2}{2R} - \frac{1+\theta}{R}L(\theta), \tag{5.17}$$

$$v_{01} = \frac{\beta_1 f}{12R(1-\nu)}\theta(1-\theta)(1-2\theta), \tag{5.18}$$

$$v_{02} = -\frac{\alpha_1}{8R}\theta^2(1-\theta)^2[\alpha_0 - 2\log 2q + \pi\cot\pi\theta]$$
$$+\frac{f\beta_1}{16R(1-\nu)}\theta(1-\theta) + \frac{\alpha_1}{16R}\theta(1-\theta^2)[11\theta^2 - 15\theta + 6 + 8(1+\theta)L(\theta)], \tag{5.19}$$

where the function $L(\theta)$ has been defined by

$$L(\theta) = \frac{1}{\Gamma(1-\theta)}\sum_{k=1}^{\infty}\frac{\Gamma(2+k-\theta)}{k\Gamma(3+k)}. \tag{5.20}$$

6. EQUILIBRIUM OF THE STAMP

If we assume that the loading forces P and fP, exerted on the stamp, act through the center of mass 0 we have the situation as shown in Fig. 2. Evidently there is equilibrium of forces in the horizontal and vertical directions, whereas a moment M must be applied to the cylinder which is equal to

$$M = fPR + \int_0^1 p(y)(d-y)\,\mathrm{d}y. \tag{6.1}$$

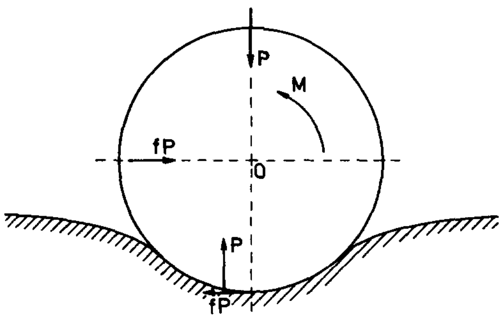

FIG. 2. Loading of the cylinder.

In the zero'th approximation (6.1) becomes

$$M_0 = \frac{\theta(1-\theta)}{12R}[3fR + 2(1-2\theta)]. \tag{6.2}$$

If the stamp is loaded by the forces P and fP, acting through 0, but the moment (6.1) is not applied, the pressure distribution $p(x)$ and the shear stress t_{xy}, as has been given by (3.8), (5.1)–(5.4) and (2.6) respectively, are not in equilibrium with the loading forces. Hence, the given solution does not hold, although it satisfies the equations and the prescribed boundary conditions. The reason for this is that in the problem under discussion we only can prescribe the normal displacement under the stamp together with the stresses outside of the stamp, but the boundary condition (2.6) has to be replaced by

$$|t_{xy}| \leq f|\sigma_y|, \tag{6.3}$$

while the horizontal force H satisfies the inequality

$$H \leq fP. \tag{6.4}$$

If we apply the forces P and fP, together with the moment M according to (6.1) the boundary condition (2.6) will be satisfied.

7. NUMERICAL RESULTS

We have computed the pressure distribution $p(x)$, the displacement v_0 and the total load P for some values of Poisson's ratio and the friction coefficient. The results have been collected in the Tables 2–4.

In Fig. 3 the pressure distribution $p(x)$ is shown for $q = 6$.

TABLE 2

		$p(x)$	
ν	x	$f = 0{\cdot}1$	$f = 0{\cdot}5$
0	0·2	$0{\cdot}650\ 10^{-1} + 0{\cdot}466\ 10^{-2}q^{-2} - 0{\cdot}148\ 10^{-3}q^{-3}$	$0{\cdot}688\ 10^{-1} + 0{\cdot}482\ 10^{-2}q^{-2} - 0{\cdot}476\ 10^{-3}q^{-3}$
	0·4	$0{\cdot}784\ 10^{-1} + 0{\cdot}562\ 10^{-2}q^{-2} - 0{\cdot}479\ 10^{-4}q^{-3}$	$0{\cdot}781\ 10^{-1} + 0{\cdot}547\ 10^{-2}q^{-2} + 0{\cdot}951\ 10^{-4}q^{-3}$
	0·6	$0{\cdot}774\ 10^{-1} + 0{\cdot}555\ 10^{-2}q^{-2} + 0{\cdot}815\ 10^{-4}q^{-3}$	$0{\cdot}733\ 10^{-1} + 0{\cdot}513\ 10^{-2}q^{-2} + 0{\cdot}685\ 10^{-3}q^{-3}$
	0·8	$0{\cdot}622\ 10^{-1} + 0{\cdot}446\ 10^{-2}q^{-2} + 0{\cdot}169\ 10^{-3}q^{-3}$	$0{\cdot}554\ 10^{-1} + 0{\cdot}388\ 10^{-2}q^{-2} + 0{\cdot}969\ 10^{-3}q^{-3}$
0·2	0·2	$0{\cdot}647\ 10^{-1} + 0{\cdot}523\ 10^{-2}q^{-2} - 0{\cdot}167\ 10^{-3}q^{-3}$	$0{\cdot}679\ 10^{-1} + 0{\cdot}542\ 10^{-2}q^{-2} - 0{\cdot}624\ 10^{-3}q^{-3}$
	0·4	$0{\cdot}783\ 10^{-1} + 0{\cdot}633\ 10^{-2}q^{-2} - 0{\cdot}579\ 10^{-4}q^{-3}$	$0{\cdot}785\ 10^{-1} + 0{\cdot}626\ 10^{-2}q^{-2} - 0{\cdot}597\ 10^{-5}q^{-3}$
	0·6	$0{\cdot}775\ 10^{-1} + 0{\cdot}627\ 10^{-2}q^{-2} + 0{\cdot}858\ 10^{-4}q^{-3}$	$0{\cdot}748\ 10^{-1} + 0{\cdot}597\ 10^{-2}q^{-2} + 0{\cdot}676\ 10^{-3}q^{-3}$
	0·8	$0{\cdot}626\ 10^{-1} + 0{\cdot}506\ 10^{-2}q^{-2} + 0{\cdot}185\ 10^{-3}q^{-3}$	$0{\cdot}577\ 10^{-1} + 0{\cdot}460\ 10^{-2}q^{-2} + 0{\cdot}105\ 10^{-2}q^{-3}$
0·3	0·2	$0{\cdot}644\ 10^{-1} + 0{\cdot}577\ 10^{-2}q^{-2} - 0{\cdot}187\ 10^{-3}q^{-3}$	$0{\cdot}671\ 10^{-1} + 0{\cdot}596\ 10^{-2}q^{-2} - 0{\cdot}763\ 10^{-3}q^{-3}$
	0·4	$0{\cdot}782\ 10^{-1} + 0{\cdot}700\ 10^{-2}q^{-2} - 0{\cdot}677\ 10^{-4}q^{-3}$	$0{\cdot}786\ 10^{-1} + 0{\cdot}698\ 10^{-2}q^{-2} - 0{\cdot}983\ 10^{-4}q^{-3}$
	0·6	$0{\cdot}777\ 10^{-1} + 0{\cdot}695\ 10^{-2}q^{-2} + 0{\cdot}912\ 10^{-4}q^{-3}$	$0{\cdot}758\ 10^{-1} + 0{\cdot}673\ 10^{-2}q^{-2} + 0{\cdot}673\ 10^{-3}q^{-3}$
	0·8	$0{\cdot}628\ 10^{-1} + 0{\cdot}562\ 10^{-2}q^{-2} + 0{\cdot}202\ 10^{-3}q^{-3}$	$0{\cdot}592\ 10^{-1} + 0{\cdot}525\ 10^{-2}q^{-2} + 0{\cdot}112\ 10^{-2}q^{-3}$
0·4	0·2	$0{\cdot}641\ 10^{-1} + 0{\cdot}664\ 10^{-2}q^{-2} - 0{\cdot}222\ 10^{-3}q^{-3}$	$0{\cdot}658\ 10^{-1} + 0{\cdot}679\ 10^{-2}q^{-2} - 0{\cdot}997\ 10^{-3}q^{-3}$
	0·4	$0{\cdot}781\ 10^{-1} + 0{\cdot}808\ 10^{-2}q^{-2} - 0{\cdot}845\ 10^{-4}q^{-3}$	$0{\cdot}785\ 10^{-1} + 0{\cdot}811\ 10^{-2}q^{-2} - 0{\cdot}260\ 10^{-3}q^{-3}$
	0·6	$0{\cdot}778\ 10^{-1} + 0{\cdot}805\ 10^{-2}q^{-2} + 0{\cdot}100\ 10^{-3}q^{-3}$	$0{\cdot}769\ 10^{-1} + 0{\cdot}793\ 10^{-2}q^{-2} + 0{\cdot}656\ 10^{-3}q^{-3}$
	0·8	$0{\cdot}631\ 10^{-1} + 0{\cdot}654\ 10^{-2}q^{-2} + 0{\cdot}232\ 10^{-3}q^{-3}$	$0{\cdot}612\ 10^{-1} + 0{\cdot}631\ 10^{-2}q^{-2} + 0{\cdot}125\ 10^{-2}q^{-3}$
0·5	0·2	$0{\cdot}637\ 10^{-1} + 0{\cdot}815\ 10^{-2}q^{-2} - 0{\cdot}286\ 10^{-3}q^{-3}$	$0{\cdot}637\ 10^{-1} + 0{\cdot}815\ 10^{-2}q^{-2} - 0{\cdot}143\ 10^{-2}q^{-3}$
	0·4	$0{\cdot}780\ 10^{-1} + 0{\cdot}998\ 10^{-2}q^{-2} - 0{\cdot}117\ 10^{-3}q^{-3}$	$0{\cdot}780\ 10^{-1} + 0{\cdot}998\ 10^{-2}q^{-2} - 0{\cdot}584\ 10^{-3}q^{-3}$
	0·6	$0{\cdot}780\ 10^{-1} + 0{\cdot}998\ 10^{-2}q^{-2} + 0{\cdot}117\ 10^{-3}q^{-3}$	$0{\cdot}780\ 10^{-1} + 0{\cdot}998\ 10^{-2}q^{-2} + 0{\cdot}584\ 10^{-3}q^{-3}$
	0·8	$0{\cdot}637\ 10^{-1} + 0{\cdot}815\ 10^{-2}q^{-2} + 0{\cdot}286\ 10^{-3}q^{-3}$	$0{\cdot}637\ 10^{-1} + 0{\cdot}815\ 10^{-2}q^{-2} + 0{\cdot}143\ 10^{-2}q^{-3}$

TABLE 3

			v_0	
ν	f	v_{00}	v_{01}	v_{02}
0	0·1	$-0{\cdot}421 - 0{\cdot}125 \log q$	$+0{\cdot}198\ 10^{-3}q^{-1}$	$-0{\cdot}625\ 10^{-1}q^{-2} - 0{\cdot}179\ 10^{-1}q^{-2} \log q$
	0·5	$-0{\cdot}467 - 0{\cdot}122 \log q$	$+0{\cdot}475\ 10^{-2}q^{-1}$	$-0{\cdot}592\ 10^{-1}q^{-2} - 0{\cdot}171\ 10^{-1}q^{-2} \log q$
0·2	0·1	$-0{\cdot}409 - 0{\cdot}125 \log q$	$+0{\cdot}135\ 10^{-3}q^{-1}$	$-0{\cdot}691\ 10^{-1}q^{-2} - 0{\cdot}202\ 10^{-1}q^{-2} \log q$
	0·5	$-0{\cdot}444 - 0{\cdot}123 \log q$	$+0{\cdot}330\ 10^{-2}q^{-1}$	$-0{\cdot}658\ 10^{-1}q^{-2} - 0{\cdot}197\ 10^{-1}q^{-2} \log q$
0·3	0·1	$-0{\cdot}397 - 0{\cdot}125 \log q$	$+0{\cdot}985\ 10^{-4}q^{-1}$	$-0{\cdot}746\ 10^{-1}q^{-2} - 0{\cdot}224\ 10^{-1}q^{-2} \log q$
	0·5	$-0{\cdot}423 - 0{\cdot}124 \log q$	$+0{\cdot}243\ 10^{-2}q^{-1}$	$-0{\cdot}709\ 10^{-1}q^{-2} - 0{\cdot}220\ 10^{-1}q^{-2} \log q$
0·4	0·1	$-0{\cdot}375 - 0{\cdot}125 \log q$	$+0{\cdot}557\ 10^{-4}q^{-1}$	$-0{\cdot}820\ 10^{-1}q^{-2} - 0{\cdot}259\ 10^{-1}q^{-2} \log q$
	0·5	$-0{\cdot}390 - 0{\cdot}125 \log q$	$+0{\cdot}139\ 10^{-2}q^{-1}$	$-0{\cdot}772\ 10^{-1}q^{-2} - 0{\cdot}257\ 10^{-1}q^{-2} \log q$
0·5	0·1	$-0{\cdot}334 - 0{\cdot}125 \log q$	$+0 \qquad q^{-1}$	$-0{\cdot}916\ 10^{-1}q^{-2} - 0{\cdot}320\ 10^{-1}q^{-2} \log q$
	0·5	$-0{\cdot}334 - 0{\cdot}125 \log q$	$+0 \qquad q^{-1}$	$-0{\cdot}816\ 10^{-1}q^{-2} - 0{\cdot}320\ 10^{-1}q^{-2} \log q$

TABLE 4

	P	
ν	$f = 0{\cdot}1$	$f = 0{\cdot}5$
0	$0{\cdot}624\ 10^{-1} + 0{\cdot}895\ 10^{-2}q^{-2} + 0{\cdot}110\ 10^{-4}q^{-3}$	$0{\cdot}610\ 10^{-1} + 0{\cdot}854\ 10^{-2}q^{-2} + 0{\cdot}258\ 10^{-3}q^{-3}$
0·2	$0{\cdot}625\ 10^{-1} + 0{\cdot}101\ 10^{-1}q^{-2} + 0{\cdot}918\ 10^{-5}q^{-3}$	$0{\cdot}616\ 10^{-1} + 0{\cdot}983\ 10^{-2}q^{-2} + 0{\cdot}221\ 10^{-3}q^{-3}$
0·3	$0{\cdot}625\ 10^{-1} + 0{\cdot}112\ 10^{-1}q^{-2} + 0{\cdot}773\ 10^{-5}q^{-3}$	$0{\cdot}620\ 10^{-1} + 0{\cdot}110\ 10^{-1}q^{-2} + 0{\cdot}189\ 10^{-3}q^{-3}$
0·4	$0{\cdot}625\ 10^{-1} + 0{\cdot}129\ 10^{-1}q^{-2} + 0{\cdot}525\ 10^{-5}q^{-3}$	$0{\cdot}623\ 10^{-1} + 0{\cdot}129\ 10^{-1}q^{-2} + 0{\cdot}130\ 10^{-3}q^{-3}$
0·5	$0{\cdot}625\ 10^{-1} + 0{\cdot}160\ 10^{-1}q^{-2} + 0 \qquad q^{-3}$	$0{\cdot}625\ 10^{-1} + 0{\cdot}160\ 10^{-1}q^{-2} + 0 \qquad q^{-3}$

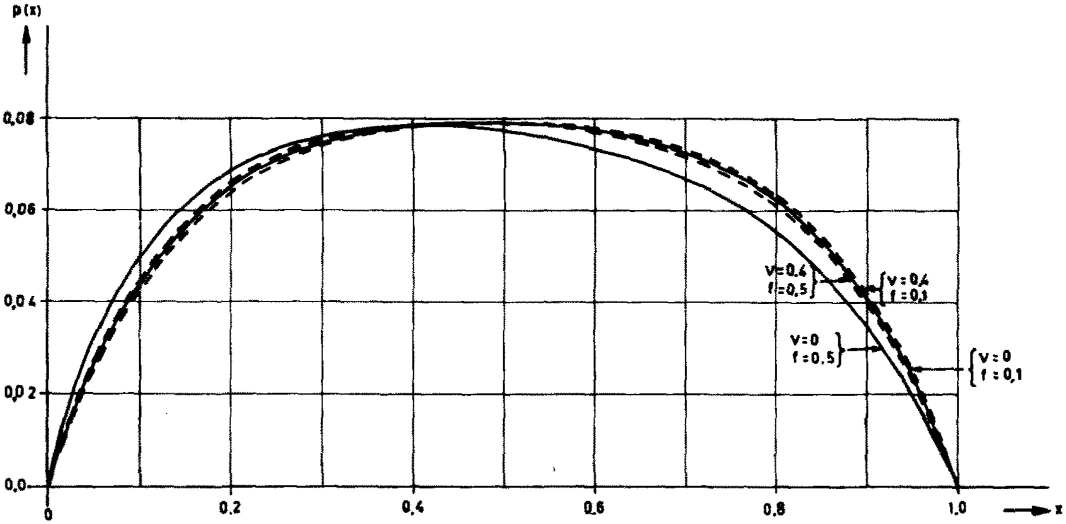

FIG. 3. The pressure p across the contact region for various values of Poisson's ratio ν and the friction f.

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Абстракт—Получается приближенные решения контактной задачи, касающейся слоя конечной толщины нагруженного шереховатым цилиндрическим штампом, который движется вдоль границы. Предлагается постоянный козффициент трения. Нижняя сторона слоя прикреплена к житкому оснрванию. В задаче пренебрагается инерционными усилиями. Решение приближается с помощью решения для плоской деформации. Такое решение выражается сходящими рядами, в степенях обратного параметра толщины, то есть, отношения величины толщыны и длины контакта. В зтом выражении козффициенты удовлетворяют сингулярным интегральным уровнениям. Пелучаются численные результаты для больших значании параметра толщины. Во второй части предлагаемого исследования будут определены асимптомики для тонкого слоя.